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High-energy limit versus infinite-mass limit in the Coulomb problem

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Abstract. Relativistic eikonal physics is applied to Coulomb scattering and bound states. It is shown that the eikonal approximation can accommodate the impact factor result as well as the non-relativistic Balmer spectrum. It is argued that relativistic corrections are outside this realm of approximation. This will be confirmed by eikonalizing the high-energy scattering amplitude of an electron in a static Coulomb field which is generated by a proton.

1. Introduction

Some time ago the classical one-particle Dirac equation was re-examined within the context of the relativistic eikonal approximation (Dittrich 1970). It was shown that the poles of a semi-eikonalized two-body scattering amplitude coincide with the well-known relativistic Balmer spectrum. The evaluation made substantial use of an infinite-mass limiting process. Beyond that equivalence proof nobody seems to have succeeded in constructing the relativistic Balmer formula in an absolutely convincing manner (Brezin *et al* 1970, Frondsal 1967, Nambu 1967, Todorov 1971). The computation of the poles of the totally eikonalized crossed ladder graphs of electrodynamics always yields instead the non-relativistic counterpart and it is hard to see how relativistic corrections of E_n which depend on both n and l (or j) could be directly derived within the eikonal approximation. It is, however, interesting to observe that the eikonal approach can reproduce Cheng and Wu's (1969a, also Chang and Ma 1969) impact factor result and at the same time allows for the non-relativistic bound state formula, that is, reveals a pole structure in a totally eikonalized scattering matrix, whose realm of validity is supposed to be $s \to \infty$, not s below the threshold in the bound state region. In fact, this non-relativistic feature is also shared by other approaches to high-energy scattering in quantum electrodynamics, eg, as expressed by the infinite-momentum results in the work of Kogut and Soper (1970). Their choice of new time and space variables (τ, z) drew attention to the two-dimensional Galilean group of non-relativistic quantum mechanics in two dimensions, which led to a non-relativistic structure for quantum mechanics in the infinite-momentum frame. We consider it therefore not merely an accident that the eikonal approach, or the infinite-momentum frame for that matter, is also reflected in the bound state realm of QED, where only the classical Balmer spectrum is projected out of the complex pole structure of the electron's Green's function in presence of a Coulomb potential.

In order to present our procedure, let us first recall some ideas and results of Dittrich (1970). Thereafter we will give an easy proof of the impact factor representation for high-energy Coulomb scattering. These results are shown to appear in the lowest-order expansion of the closed-form expression for the eikonalized scattering amplitude of an electron in the static Coulomb field.

2. Dirac equation and subclass of Feynman diagrams

In order to investigate the two-particle reaction (figure 1), $p_1+p_2 \rightarrow p'_1+p'_2$, we start out with the following qualifications: let the top line (1) represent a fermion (electron) and the bottom line (2) a heavy scalar particle (nucleus) which can form bound states. Furthermore, the coordinate system is chosen so that the electron is moving in the z direction.

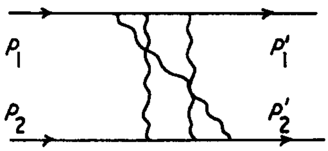

Figure 1. Multiphotqn exchange process

Following the standard *LSZ* reduction techniques (Fried 1972), the corresponding S-matrix element may be written in the form

$$\langle p'_2, p'_1|S|p_1, p_2\rangle$$
$$= \langle p'_2|_{\mathrm{IN}}\langle 0|b_{\mathrm{IN}}(p'_1) : \exp\left[\int \left(\bar{\psi}_{\mathrm{IN}}\overrightarrow{\mathscr{D}}\frac{\delta}{\delta\bar{\eta}} - \frac{\delta}{\delta\eta}\overleftarrow{\mathscr{D}}\psi_{\mathrm{IN}}\right)\right] : b^{+}_{\mathrm{IN}}(p_1)|0\rangle_{\mathrm{IN}}|p_2\rangle$$
$$\times z|_{\eta=\bar{\eta}=0} \tag{2.1}$$

where the generating functional z is simply

$$z\{\eta, \bar{\eta}\} = \exp(-\mathrm{i}\bar{\eta}G[A]\eta) \tag{2.2}$$

and the electron's Green's function is defined by

$$[\gamma^{\mu}(\mathrm{i}\,\partial_{\mu}-eA_{\mu})-m]G(x'_1, x_1|A) = \delta^4(x'_1-x_1). \tag{2.3}$$

$A_{\mu}(x)$ represents the external c-number field which is generated by the heavy scalar nucleus. The $\eta, \bar{\eta}$ denote artificial anticommuting c-number sources and the free Dirac operator $\mathscr{D}_x$ is conventionally defined via

$$(\mathrm{i}\gamma\,\partial_x - m)S_F(x-y) : = \overrightarrow{\mathscr{D}}_x S_F(x-y) = \delta^4(x-y).$$

Equation (2.1) can be further reduced to

$$\langle p'_2, p'_1|S|p_1, p_2\rangle$$
$$= -\frac{1}{(2\pi)^3}\left(\frac{m^2}{E(p_1)E(p'_1)}\right)^{1/2}\int \mathrm{d}^4x_1\,\mathrm{d}^4x'_1\,\exp[\mathrm{i}(p_1x'_1-p_1x_1)]$$
$$\times\,\bar{u}(s'_1, p_1)\overrightarrow{\mathscr{D}}_{x'_1}\langle p'_2|T(\psi(x'_1)\bar{\psi}(x_1))|p_2\rangle\overleftarrow{\mathscr{D}}_{x_1}u(s_1, p_1). \tag{2.4}$$

The object of interest is therefore

$$p_2' = p_2+q : \langle p_2+q|T(\psi(x_1')\bar{\psi}(x_1))|p_2\rangle$$
$$\equiv \langle p_2+q|\mathrm{i}G(x_1', x_1|A)|p_2\rangle = \frac{\mathrm{i}}{(2\pi)^3}\exp(\mathrm{i}qx_1')\int\frac{\mathrm{d}^4\Delta}{(2\pi)^4}$$
$$\times\exp[\mathrm{i}\,\Delta(x_1'-x_1)]G(\Delta) \tag{2.5}$$

and Δ is fixed by $(p_2+\Delta)^2 = m_B^2$.

We intend to employ the eikonal approximation for the scalar particle which means that the matrix element $\langle p_2+q|T(\psi(x_1')\bar{\psi}(x_1))|p_2\rangle$ can be further reduced. Omitting radiative corrections also on the scalar particle, the contribution of the latter to the S-matrix is then stated by

$$_{\mathrm{IN}}\langle 0|a_{\mathrm{IN}}(p_2') : \exp\left[\int.\left(\phi_{\mathrm{IN}}^+ K\frac{\delta}{\delta j^+}+\frac{\delta}{\delta j}K\phi_{\mathrm{IN}}\right)\right] : a_{\mathrm{IN}}^+(p_2)|0\rangle_{\mathrm{IN}}\exp(-\mathrm{i}j^+\,\Delta[A]j)|_{j=j^+=0}$$
$$= \frac{1}{(2\pi)^3}\left[\frac{1}{2\omega(p_2)2\omega(p_2')}\right]^{1/2}\int\mathrm{d}^4x_2\,\mathrm{d}^4x_2'\exp[\mathrm{i}(p_2'x_2'-p_2x_2)]$$
$$\times K_{x_2'}(-\mathrm{i}\,\Delta(x_2', x_2|A))K_{x_2}.$$

The Klein–Gordon operator K_x satisfies the Green's function equation

$$(\partial^2+M^2)\,\Delta_F(x) = K_x\,\Delta_F(x) = -\delta^4(x)$$

which, in presence of an external potential $A_\mu(x)$, turns into

$$[(\mathrm{i}\,\partial_\mu - eA_\mu)^2 - M^2]\,\Delta(x, y|A) = \delta^4(x-y).$$

Connection of the electron and scalar particle via photon exchange yields the complete S-matrix element:

$$_{\mathrm{IN}}\langle p_1', p_2'|S|p_1, p_2\rangle_{\mathrm{IN}}$$
$$= -\frac{1}{(2\pi)^6}\left[\frac{m^2}{E(p_1)E(p_1')2\omega(p_2)2\omega(p_2')}\right]^{1/2}\int\mathrm{d}^4x_1\ldots\mathrm{d}^4x_2'$$
$$\times\exp[\mathrm{i}(p_1'x_1'+p_2'x_2'-p_1x_1-p_2x_2)](\bar{u}(s_1', p_1')\vec{\mathscr{D}}_{x_1'})K_{x_2'}\exp\left(\frac{\mathrm{i}}{2}\frac{\delta}{\delta A}D_F\frac{\delta}{\delta A}\right)$$
$$\times[G(x_1', x_1|A)\,\Delta(x_2', x_2|A]|_{A=0}K_{x_2}(\overleftarrow{\mathscr{D}}_{x_1}u(s_1, p_1)). \tag{2.6}$$

Using the 'linkage formula' (Fried 1972), ie,

$$\exp\left(\frac{\mathrm{i}}{2}\frac{\delta}{\delta A}D_F\frac{\delta}{\delta A}\right)G[A]\,\Delta[A]$$
$$= \left(\exp\left(\frac{\mathrm{i}}{2}\frac{\delta}{\delta A_1}D_F\frac{\delta}{\delta A_1}\right)G[A_1]\right)\exp\left(\mathrm{i}\frac{\overleftarrow{\delta}}{\delta A_1}D_F\frac{\vec{\delta}}{\delta A_2}\right)$$
$$\times\left(\exp\left(\frac{\mathrm{i}}{2}\frac{\delta}{\delta A_2}D_F\frac{\delta}{\delta A_2}\right)\Delta[A_2]\right)$$

and restricting ourselves to undressed particles, we are now in a position to compute

$$T(x'_1, x_1|x'_2, x_2)$$
$$= \exp\left(\mathrm{i}\frac{\delta}{\delta A_1} D_F \frac{\delta}{\delta A_2}\right) G(x'_1, x_1|A_1)\, \Delta(x'_2, x_2|A_2)|_{A_1=A_2=0}. \tag{2.7}$$

In this subsection we are mainly interested in a closed-form solution of the Green's function equation for the scalar particle. This solution can be located, eg, in Dittrich (1970). There we found for the amputated eikonalized propagator

$$\Delta_{\mathrm{Eik}}(\bar{p}'_2, \bar{p}_2|A)$$
$$= 2M\mathrm{i}\int \mathrm{d}^4x \exp[\mathrm{i}(p'_2 - p_2)x]$$
$$\times\left\{\frac{\partial}{\partial s}\exp\left[-\frac{\mathrm{i}}{2M}\int_{-\infty}^{s} \mathrm{d}s' 2ep_2^{\mu} A_{\mu}(x - p_2/Ms')\right]\right\}_{s=0}. \tag{2.8}$$

If we then compare equations (2.4) and (2.6) we are led to

$$\langle p'_2|T(\psi(x'_1)\bar{\psi}(x_1))|p_2\rangle$$
$$= \frac{1}{(2\pi)^3}\left[\frac{1}{2\omega(p_2)2\omega(p'_2)}\right]^{1/2}$$
$$\times \Delta_{\mathrm{Eik}}\left(\bar{p}'_2, \bar{p}_2|\mathrm{i}\int D_F(z - v)\frac{\delta}{\delta A_1(v)}\right) G(x'_1, x_1|A_1)|_{A_1=0}.$$

Substituting formula (2.8) in (2.9) yields, after some lines of calculation,

$$\langle p_2 + q|T(\psi(x'_1)\bar{\psi}(x_1))|p_2\rangle$$
$$= \frac{\mathrm{i}}{(2\pi)^3}\exp(\mathrm{i}qx'_1)\int \frac{\mathrm{d}^4\Delta}{(2\pi)^4}\exp[-\mathrm{i}\,\Delta(x'_1 - x)]G(\Delta|A)$$
$$= \mathrm{i}2M\frac{1}{(2\pi)^3}\left[\frac{1}{2\omega(p_2)2\omega(p'_2)}\right]^{1/2}\int \mathrm{d}^3z\, \exp(\mathrm{i}qz)U\left[L\left(\frac{p_2}{M}\right)\right]\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{d}^4k'}{(2\pi)^4}$$
$$\times (2\pi)\delta^{(1)}(k_0 - k'_0)\exp[-\mathrm{i}kL^{-1}(x'_1 - z)]\exp[\mathrm{i}k'L^{-1}(x_1 - z)]$$
$$\times G\left(k_0; \boldsymbol{k}, \boldsymbol{k}'|A^{\mathrm{Coul}}(\xi) = \frac{e}{4\pi}\frac{1}{|\boldsymbol{\xi}|}\right)U^{-1}\left[L\left(\frac{p_2}{M}\right)\right] \tag{2.10}$$

where the Lorentz boost $L^{\mu\nu}$ is defined by $L^{\mu\nu}(p_2/M)u_\nu = p_2^{\mu}/M$ and u_ν is purely time-like, ie, $u_\nu = (1, \vec{0})$. Equation (2.10) can now be solved with respect to $G(\Delta)$. The result is stated in the following expression (Dittrich 1970):

$$G(\Delta_0, \vec{\Delta}) = 2M\left[\frac{1}{2\omega(p_2)2\omega(p'_2)}\right]^{1/2}\frac{M}{p_2^0}$$
$$\times UG_{\mathrm{Coul}}[(L^{-1})_{0\nu}(\Delta - q)^{\nu}; (L^{-1})_{m\nu}(\Delta - q)^{\nu}, (L^{-1})_{m\nu}\Delta^{\nu}]U.$$

The quantity of interest, the shifted Coulomb energy Δ_0, is then computed and gives the remarkably simple relation

$$\Delta_0 = (L^{-1})_{0\nu}(\Delta - q)^{\nu} = \frac{1}{M}(p_2^0\,\Delta^0 - p_2^z\,\Delta^z).$$

Using this relation in $(\Delta'+p_2)^2 = m_B^2$ we obtain $m_B = M+\Delta'_0 = M-E_n$ where Δ'_0 now coincides with the relativistic binding energy of the electron in a Coulomb field. However, it is essential to assume $M \to \infty$. Therefore not an infinite momentum-limiting process $p_z \to \infty$ (or $s \to \infty$) is the adequate method to generate the relativistic Balmer spectrum, but a simple assumption for the particle in which the electron travels must be made: it just has to be heavy.

3. Impact factor representation in high-energy Coulomb scattering

Equation (2.1) sets the stage for computing the various Feynman graphs for electron scattering by an external potential. To second order in e one finds

$$\begin{aligned}
&{}_{\rm IN}\langle p'_1, s'_1|S^{(2)}|p_1, s_1\rangle_{\rm IN} \\
&\quad = -\frac{\mathrm{i}e^2}{(2\pi)^3}\left[\frac{m^2}{E(p_1)E(p'_1)}\right]^{1/2}\int\frac{\mathrm{d}^4q}{(2\pi)^4}\bar{u}(s'_1, p'_1)\gamma^\mu A_\mu(p'_1-q) \\
&\quad \times\frac{1}{\gamma q-m}\gamma^\nu A_\nu(q-p_1)u(s_1, p_1).
\end{aligned}\tag{3.1}$$

Expression (3.1) was exactly calculated by Dalitz (1951) for the static Coulomb potential, where

$$\gamma^\mu A_\mu(q) = \gamma^0 A_0(q) = 2\pi\delta(q_0)\frac{-Ze\gamma_0}{\mu^2+|\boldsymbol{q}|^2}.$$

It is convenient to rewrite (3.1) in the following variables:

$$p'_1-p_1 = k, \qquad p'_1+p_1 = 2p$$

or

$$p'_1 = p+\frac{k}{2}, \qquad p_1 = p-\frac{k}{2}.$$

On the mass shell we have $p^2+(k/2)^2 = m^2, pk = 0$. In terms of these variables the second-order process (figure 2) leads to

$$M_{fi}^{(2)} = e^2\int\frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3}\bar{u}_f\frac{-Ze\gamma_0}{(k/2-\boldsymbol{q})^2+\mu^2}\,\frac{\gamma(p+q)+m}{(p+q)^2-m^2}\,\frac{-Ze\gamma_0}{(k/2+\boldsymbol{q})^2+\mu^2}u_i \tag{3.2}$$

where M is defined by

$$S_{fi} = -\mathrm{i}(\text{kin. factors})M_{fi}\begin{cases}(2\pi)\delta(E_f-E_i) & \text{static case,}\\ (2\pi)^4\delta^4(p_f-p_i) & \text{otherwise.}\end{cases}$$

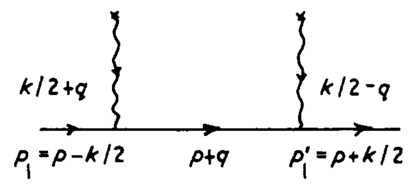

Figure 2. Second-order Coulomb process

The complicated integral (3.2) can be easily calculated in the high-energy limit $\omega \sim |\boldsymbol{p}_1| \to \infty$, where $\omega = (\boldsymbol{p}_1^2+m^2)^{1/2}$ and k fixed ($t = k^2$). One can then prove that the momentum transfer k has only transverse components lying in the (x, y) plane if the incident beam is along the z axis. It is also straightforward to show that the spin-dependent part of equation (3.2) contributes

$$\bar{u}_f\gamma_0(\gamma p+\gamma q+m)\gamma_0 u_i \sim 2\frac{\omega^2}{m}\delta_{fi} \qquad \text{as } \omega \to \infty. \tag{3.3}$$

Hence we obtain ($\frac{1}{2}k_0 = 0 = q_0$)

$$M_{fi}^{(2)} \sim e^2(-Ze)^2 2\frac{\omega^2}{m}\delta_{fi}\int\frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3}\frac{1}{\left[\left(\dfrac{\boldsymbol{k}}{2}-\boldsymbol{q}\right)^2+\mu^2\right]\left[\left(\dfrac{\boldsymbol{k}}{2}+\boldsymbol{q}\right)^2+\mu^2\right]}\frac{1}{(p+q)^2-m^2+\mathrm{i}\epsilon}$$

$$= e^2(-Ze)^2 2\frac{\omega^2}{m}\delta_{fi}\int\frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3}\frac{1}{\left[\left(\dfrac{\boldsymbol{k}}{2}-\boldsymbol{q}\right)^2+\mu^2\right]\left[\left(\dfrac{\boldsymbol{k}}{2}+\boldsymbol{q}\right)^2-\mu^2\right]}$$

$$\times\frac{1}{2}\left[\frac{1}{(p+q)^2-m^2+\mathrm{i}\epsilon}+\frac{1}{(p-q)^2-m^2+\mathrm{i}\epsilon}\right].$$

Utilizing $q_0 = 0$ and

$$(p+q)^2 = m^2-\frac{t}{4}+q^2\pm 2pq \sim \mp 2\omega q_3+m^2$$

we have in the high-energy limit

$$\left[\frac{1}{(p+q)^2-m^2+\mathrm{i}\epsilon}+\frac{1}{(p-q)^2-m^2+\mathrm{i}\epsilon}\right] \sim \left(\frac{1}{-2\omega q_3+\mathrm{i}\epsilon}+\frac{1}{2\omega q_3+\mathrm{i}\epsilon}\right) = -\mathrm{i}\frac{\pi}{\omega}\delta(q_3).$$

Finally we obtain

$$M_{fi}^{(2)} \sim -e^2(Ze)^2\mathrm{i}\frac{\omega}{2m}\delta_{fi}\int\frac{\mathrm{d}^2\boldsymbol{q}_T}{(2\pi)^2}\frac{1}{\left[\left(\boldsymbol{q}_T+\dfrac{\boldsymbol{k}_T}{2}\right)^2+\mu^2\right]\left[\left(\boldsymbol{q}_T-\dfrac{\boldsymbol{k}_T}{2}\right)^2+\mu^2\right]} \tag{3.5}$$

which means that the electron interacts with the Coulomb field through interchanging purely transverse momenta. Equation (3.5) can be cast into a form known as impact factor representation (Cheng and Wu 1969b) for forward scattering of an electron in the static field of the Coulomb potential:

$$M_{eN}^{(2)} \sim -\mathrm{i}(2\omega M)\int\frac{\mathrm{d}^2\boldsymbol{q}_T}{(2\pi)^2}\frac{1}{\left[\left(\boldsymbol{q}_T+\dfrac{\boldsymbol{k}_T}{2}\right)^2+\mu^2\right]\left[\left(\boldsymbol{q}_T-\dfrac{\boldsymbol{k}_T}{2}\right)^2+\mu^2\right]}\mathscr{I}^e\mathscr{I}^N, \tag{3.6}$$

where $\mathscr{I}^e$ and $\mathscr{I}^N$ are the 'impact factors' for the electron and the nucleus, respectively:

$$\mathscr{I}^e = \frac{1}{2}\frac{e^2}{m}\delta_{11'}, \qquad \mathscr{I}^N = \frac{1}{2}\frac{Ze^2}{M}.$$

M is the mass of the static nucleus and ω is the energy of the electron in the laboratory frame. At this stage one would wish to push the calculation beyond equation (3.6). However, it is a rather hopeless task to try to compute the nth-order amplitude by proceeding the same way which led us to (3.6). However, a way out of this difficulty will be provided in the next section, where we will eikonalize both the electron and the proton line, which can be thought of as the generator of the Coulomb potential.

4. Eikonalization of electron–nucleus (proton) scattering

The foregoing calculation is instructive in that it provides us with a first non-trivial information in understanding the mechanism of high-energy electron–nucleus scattering. On the other hand we will show that formula (3.6) is nothing but a low-order term in the expansion of a Glauber-type eikonal formula, which we now want to exhibit.

The idea is to approximate both top and bottom line of figure 1 by their eikonal propagators. Those propagators were derived and used in the past by several authors (Abarbanel and Itzykson 1969, Dittrich 1972). Here it suffices to recall that the amputated electron's Green's function is given by

$$G_{\mathrm{Eik}}(\bar{p}_1', \bar{p}_1|A) = \mathrm{i}\int \mathrm{d}^4x\, \exp[\mathrm{i}(p_1'-p_1)x]\left\{\frac{\partial}{\partial s}\exp\left[-\mathrm{i}e\int_{-\infty}^{s}\mathrm{d}s'\frac{p_1^{\mu}}{m}A_{\mu}\left(x-\frac{p_1^{\mu}}{m}s'\right)\right]\right\}_{s=0} \qquad (4.1)$$

while the corresponding scalar particle line is approximated by $\Delta_{\mathrm{Eik}}[A]$ as expressed in equation (2.8). Recalling the matrix element (2.6) which takes into account all ladder-type photon-exchange process, we observe that the quantity of interest is

$$\begin{aligned}
&\mathrm{i}(2\pi)^4\delta^4(p_1'+p_2'-p_1-p_2)M(s,t)\\
&\quad= \delta_{\lambda_1\lambda_1'}\exp\left(\mathrm{i}\frac{\delta}{\delta A_1}D_F\frac{\delta}{\delta A_2}\right)\Delta_{\mathrm{Eik}}(\bar{p}_2', p_2|A_2)G_{\mathrm{Eik}}(\bar{p}_1', \bar{p}_1|A_1)|_{A_1=A_2=0}\\
&\quad= \delta_{\lambda_1\lambda_1'}(2M\mathrm{i})\mathrm{i}(2\pi)^4\delta^4(p_1'+p_2'-p_1-p_2)\int \mathrm{d}^4\xi\,\exp[-\mathrm{i}(p_1'-p_1)\xi]\\
&\qquad\times\left(\frac{\partial}{\partial s_1}\frac{\partial}{\partial s_2}\exp\left[-\mathrm{i}e_1e_2\left(\frac{p_1}{m}\right)\left(\frac{p_2}{M}\right)\int_{-\infty}^{s_1}\mathrm{d}s_1'\int_{-\infty}^{s_2}\mathrm{d}s_2'\right.\right.\\
&\qquad\times\left.\left. D_F\left(-\xi-\frac{p_1}{m}s_1'+\frac{p_2}{M}s_2'\right)\right]\right)_{s_1=s_2=0}.
\end{aligned}$$

The resulting equation for $M(s,t)$ is therefore

$$\begin{aligned}
M(s,t) &= 2M\mathrm{i}\delta_{\lambda_1\lambda_1'}\int \mathrm{d}^4\xi\,\exp[-\mathrm{i}(p_1'-p_1)\xi]\\
&\quad\times\left\{\frac{\partial}{\partial s_1}\frac{\partial}{\partial s_2}\exp\left[-\mathrm{i}e_1e_2\left(\frac{p_1}{m}\right)\left(\frac{p_2}{M}\right)\int_{-\infty}^{s_1}\mathrm{d}s_1'\int_{-\infty}^{s_2}\mathrm{d}s_2'\right.\right.\\
&\quad\left.\left. D_F\left(-\xi-\frac{p_1}{m}s_1'+\frac{p_2}{M}s_2'\right)\right]\right\}_{s_1=s_2=0}.
\end{aligned} \qquad (4.2)$$

Equation (4.2) is most conveniently computed in the CM system where $\boldsymbol{p}_1+\boldsymbol{p}_2=0$. If $\boldsymbol{p}_1$ is taken in the z direction, $|\boldsymbol{p}_1|=p_z$, we have

$$p_1^0+p_2^0=\sqrt{s}, \qquad p_1p_2=\frac{s-m^2-M^2}{2}$$

and

$$|\boldsymbol{p}_1|=\frac{1}{2\sqrt{s}}[s-(m+M)^2]^{1/2}[s-(m-M)^2]^{1/2}.$$

A decomposition of ξ^μ in

$$\xi^\mu=\xi_T^\mu+\xi_1\frac{p_1^\mu}{m}+\xi_2\frac{p_2^\mu}{M}$$

where ξ_T^μ has nonvanishing components only in the xy plane, yields

$$M(s,t)=2Mi\delta_{\lambda_1\lambda_1'}\frac{(p_1^0+p_2^0)}{mM}|\boldsymbol{p}_1|\int d^2\boldsymbol{\xi}_T\exp[i(\boldsymbol{p}_1'-\boldsymbol{p}_1)_T\cdot\boldsymbol{\xi}_T]\int_{-\infty}^{+\infty}d\xi_1\int_{-\infty}^{+\infty}d\xi_2$$
$$\times\left(\frac{\partial}{\partial s_1}\frac{\partial}{\partial s_2}\exp\left\{-ie_1e_2\frac{p_1p_2}{mM}\int_{-\infty}^{s_1}ds_1'\int_{-\infty}^{s_2}ds_2'\right.\right.$$
$$\left.\left.\times D_F\left[-\xi_T^\mu-\frac{p_1^\mu}{m}(\xi_1+s_1')+\frac{p_2^\mu}{M}(-\xi_2+s_2')\right]\right\}\right)_{s_1=s_2=0}.$$

A change of variables then implies the following form for $M(s,t)$:

$$M(s,t)=2Mi\delta_{\lambda_1\lambda_1'}\frac{(p_1^0+p_2^0)}{mM}|\boldsymbol{p}_1|\int d^2\xi_T\exp(i\boldsymbol{k}_T\cdot\boldsymbol{\xi}_T)\int d\xi_1\int d\xi_2$$
$$\times\left\{\frac{\partial}{\partial\xi_1}\frac{\partial}{\partial\xi_2}\exp\left[-ie_1e_2\frac{p_1p_2}{mM}\int_{-\infty}^{\xi_1}d\sigma_1\int_{-\infty}^{\xi_2}d\sigma_2\right.\right.$$
$$\left.\left.\times D_F\left(-\xi_T^\mu-\sigma_1\frac{p_1^\mu}{m}+\sigma_2\frac{p_2^\mu}{M}\right)\right]\right\}=2Mi\delta_{\lambda_1\lambda_1'}\frac{(p_1^0+p_2^0)}{mM}|\boldsymbol{p}_1|\int d^2\boldsymbol{\xi}_T$$
$$\times\exp(i\boldsymbol{k}_T\boldsymbol{\xi}_T)\left(\exp\left[-ie_1e_2(p_1p_2)\int_{\infty}^{+\infty}d\tau_1\int_{-\infty}^{+\infty}d\tau_2\right.\right.$$
$$\left.\left.\times D_F(-\xi_T^\mu-p_1^\mu\tau_1+p_2^\mu\tau_2)\right]-1\right).$$

Employing the Fourier transform of $D_F(x)$, ie,

$$D_F(x)=\int\frac{d^4k}{(2\pi)^4}\exp(-ikx)\frac{-1}{k^2-\mu^2+i\epsilon}$$

we need to compute

$$\frac{1}{(2\pi)^4}\int d^4k\int d\tau_1\int d\tau_2\exp(ik_\mu\xi_T^\mu+ik_\mu p_1^\mu\tau_1-ik_\mu p_2^\mu\tau_2)\frac{1}{-k^2+\mu^2-i\epsilon}$$
$$=\frac{1}{|\boldsymbol{p}_1|(p_1^0+p_2^0)}\int\frac{d^2\boldsymbol{k}_T}{(2\pi)^2}\frac{\exp(-i\boldsymbol{k}_T\cdot\boldsymbol{\xi}_T)}{\boldsymbol{k}_T^2+\mu^2}=\frac{1}{|\boldsymbol{p}_1|(p_1^0+p_2^0)}\frac{1}{(2\pi)}K_0(\mu\xi_T). \tag{4.3}$$

The resulting equation for the scattering amplitude is therefore

$$M(s,t) = 2M\mathrm{i}\delta_{\lambda_1\lambda_1'}\frac{(p_1^0+p_2^0)}{mM}|\boldsymbol{p}_1|\int \mathrm{d}^2\xi_T \exp(\mathrm{i}\boldsymbol{k}_T\,.\,\xi_T)$$
$$\times\left(\exp\left[-\mathrm{i}e_1e_2\gamma(s)\frac{1}{2\pi}K_0(\mu\xi_T)\right]-1\right) \tag{4.4}$$

where

$$t = -|\boldsymbol{k}_T|^2 \qquad \text{and} \qquad \gamma(s) = \frac{s-m^2-M^2}{[s-(m+M)^2]^{1/2}[s-(m-M)^2]^{1/2}}. \tag{4.5}$$

Equation (4.4) contains both the impact factor result for electron nucleus scattering ($s \to \infty$, t fixed) and at the same time has in it all the information necessary to discuss bound state problems ($M \to \infty$). For a static nucleus there is no factor $2M$ in front of equation (4.4). Furthermore, in the limit $s \to \infty$, $\gamma(s) \to 1$; therefore

$$M(s,t) \underset{s\to\infty}{\simeq} \mathrm{i}\delta_{\lambda_1\lambda_1'}\frac{\omega}{m}\int \mathrm{d}^2\xi_T \exp(\mathrm{i}\boldsymbol{k}_T\,.\,\xi_T)\left\{\exp\left[-\mathrm{i}e_1e_2\frac{1}{2\pi}K_0(\mu\xi_T)\right]-1\right\} \tag{4.6}$$

where $s = 2M\omega$ and ω denotes the electron energy with respect to the rest frame of the nucleus. Considered for $e_1 = e$, $e_2 = -Ze$, equation (4.6) turns into

$$M(s,t) = \mathrm{i}\delta_{\lambda_1\lambda_1'}\frac{\omega}{m}\int \mathrm{d}^2\xi_T \exp(\mathrm{i}\boldsymbol{k}_T\,.\,\xi_T)\left\{\exp\left[\mathrm{i}Ze^2\frac{1}{2\pi}K_0(\mu\xi_T)\right]-1\right\} \tag{4.7}$$

which, when expanded in Ze^2 yields in lowest order

$$M^{(1)}(s,t) = Ze^2\frac{\omega}{m}\frac{1}{t-\mu^2}\delta_{\lambda_1\lambda_1'}$$

and

$$M^{(2)}(s,t) = -\mathrm{i}(2\omega M)\int\frac{\mathrm{d}^2\boldsymbol{q}_T}{(2\pi)^2}\frac{1}{\left[\left(\boldsymbol{q}_T+\frac{\boldsymbol{k}_T}{2}\right)^2+\mu^2\right]\left[\left(\boldsymbol{q}_T-\frac{\boldsymbol{k}_T}{2}\right)^2+\mu^2\right]}\frac{e^2}{2m}\delta_{\lambda_1\lambda_1'}\frac{Z^2e^2}{2M}.$$

Here we recognize the former impact factor result (3.6).

In QED we have to take the limit $\mu \to 0$ in equation (4.7). For $\mu \to 0$ with $\boldsymbol{k}_T$ fixed at a nonzero value, the -1 term in (4.7) does not contribute. The remaining integral of interest is then given by

$$I = \int \mathrm{d}^2\xi_T \exp(\mathrm{i}\boldsymbol{k}_T\,.\,\xi_T)\exp\left[\mathrm{i}Ze^2\frac{1}{2\pi}K_0(\mu\xi_T)\right].$$

Using the representation

$$\int_0^{2\pi}\exp(\mathrm{i}k_T\xi_T\cos\phi)\,\mathrm{d}\phi = 2\pi J_0(k_T\xi_T)$$

and the asymptotic form

$$K_0(z) = -\left(\ln\left(\frac{z}{2}\right)+\gamma\right)\left(1+\frac{z^2}{4}+\dots\right)+\dots$$

we obtain (with the aid of Bateman's Tables of Integral Transforms, vol. 2),

$$
\begin{aligned}
I \sim 2\pi \int_0^\infty & \mathrm{d}\xi_T \xi_T J_0(k_T\xi_T)(\tfrac{1}{2}\mu\xi_T)^{-\mathrm{i}Ze^2/(2\pi)} \\
&= 2\pi k_T^{-1/2}\left(\frac{\mu}{2}\right)^{-\mathrm{i}Ze^2/(2\pi)} \int_0^\infty \mathrm{d}\xi_T \xi_T^{1/2-\mathrm{i}Ze^2/(2\pi)} J_0(\xi_T k_T)(\xi_T k_T)^{1/2} \\
&= 4\pi \frac{\Gamma(1-\mathrm{i}Z\alpha)}{\Gamma(\mathrm{i}Z\alpha)} \mu^{-2\mathrm{i}Z\alpha}(\boldsymbol{k}_T^2)^{-1+\mathrm{i}Z\alpha}, \qquad \alpha = \frac{e^2}{4\pi}.
\end{aligned}
\tag{4.8}
$$

$\mu^{-2\mathrm{i}Z\alpha}$ is the infinite phase shift associated with the Coulomb field. When the cross section is computed it drops out. The factorization property of the right-hand-side of equation (4.8) is of some significance itself. The conjecture of Dalitz (1951), which states that the divergence, as $\mu \to 0$, can be factored out by a single phase factor, has been proved here to all orders in $(Z\alpha)$. This important result of eikonal physics seems to have been overlooked in the vast literature on the subject.

Going back to formula (4.4), we find bound states, since the resulting Regge-like amplitude exhibits poles when s is continued below threshold $(M+m)^2$. Like before, it is now straightforward to show that

$$
\begin{aligned}
\tilde{I} \sim (2\pi)\left(\frac{\mu}{2}\right)^{-\mathrm{i}Ze^2/(2\pi)\gamma(s)} & \int_0^\infty \mathrm{d}\xi_T \xi_T^{1-\mathrm{i}Ze^2/(2\pi)\gamma(s)} J_0(k_T\xi_T) \\
&= \frac{4\pi}{\mu^2}\left(\frac{-t}{\mu^2}\right)^{\mathrm{i}Z\alpha\gamma(s)-1} \frac{\Gamma(1-\mathrm{i}Z\alpha\gamma(s))}{\Gamma(\mathrm{i}Z\alpha\gamma(s))}
\end{aligned}
$$

which shows Regge behaviour with trajectory function $\sim\gamma(s)$. The same ratio of Γ functions appears in the exact solution of the Coulomb scattering problem. Hence the eikonalized scattering amplitude (4.4) displays poles whenever

$$
\mathrm{i}Z\alpha\gamma(s) = n, \qquad n \geqslant 1 \tag{4.9}
$$

which, upon inserting $\gamma(s)$ of equation (4.5), yields

$$
s_n = m^2 + M^2 + 2M \frac{m}{\left[1+\dfrac{(Z\alpha)^2}{n^2}\right]^{1/2}}. \tag{4.10}
$$

If we take $M \to \infty$ and define $E_n = \sqrt{s_n} - M$, we find

$$
E_n = \lim_{M\to\infty} \frac{1}{2M}(s_n - M^2) = \frac{m}{\left[1+\dfrac{(Z\alpha)^2}{n^2}\right]^{1/2}}. \tag{4.11}
$$

Evidently, this formula yields the non-relativistic (Bohr) levels.

One might wonder about the occurrence of the familiar pole-structure (4.11). After all, the eikonal method was rediscovered to investigate scattering amplitudes for large s (see Cheng and Wu 1969b, Abarbanel and Itzykson 1969). This is, however, not the only realm where eikonal methods have proved to be useful. This is due to the fact that the eikonal approximation, or in this context also known as Bloch–Nordsieck approximation, means first of all the replacement of the quantized radiation field of the exchanged photon by a classical c-number source. It is exactly this soft-photon replacement that went into the derivation of equation (4.4) and which in the sequel was discussed for the two limiting processes $s \to \infty$ and $M \to \infty$. For the appearance of bound states in the latter case we can present the following heuristic argument. The generation of bound states within a scattering problem is a non-perturbative problem which involves the summation of an infinite number of ladder and crossed-ladder type graphs. Having derived formula (4.11) one can now conclude that the main contribution to binding arises from the low-frequency spectrum of the potential generated by the heavier of the two particles (large mass M), ie, the long-wavelength part dominates this region of energy. Precisely these classical modes have been summed up in the eikonal approximation, and hence we expect to end up in the low-energy region. Surprising seems to be the fact that the *exact* Balmer spectrum emerges.

This result does not depend on the nature of the travelling particles under discussion, ie, we obtain the same bound state formula for scalar and spin $\frac{1}{2}$ carrying particles. This unpleasant feature, however, reflects the true realm of validity of eikonal physics. There is no consistent way to incorporate spin-dependent corrections which may yield the exact relativistic bound state spectrum as derived in section 2 where we just eikonalized the proton line while keeping the electron's Green's function in its exact form.

5. Conclusion

The main goal of this article was to investigate the consequences of two different limiting processes in the Coulomb problem; the high-energy limit (impact factor representation) and the infinite mass limit (energy spectrum). We have shown that the eikonal approximation can accommodate both the impact factor result and the Balmer spectrum. Interestingly enough, the eikonal approximation also yields a simple and compact answer to Dalitz's old problem concerning the divergence of the long-range Coulomb potential: not only to second order but to all orders in $Z\alpha$ can the divergence be summarized in a single phase factor.

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